IMPOSSIBILITY THEOREM FOR TWO-TIER ELECTORAL SYSTEMS

SEBASTIAN TIM HOLDUM AND FREDERIK RAVN KLAUSEN

Abstract. Two-tier electoral systems aim to balance regional representation and proportionality. Recently, the mathematical mechanisms of several Northern European two-tier systems have been challenged by increasing political party fragmentation, as evidenced by the expanding German parliament and the majority-deciding disproportionality of Denmark's 2022 election. Regionality, proportionality, and two-tier systems are defined mathematically and used to identify the common cause behind these developments through the formulation of an impossibility theorem: no electoral system with fixed parliament size can guarantee both proportional and geographic representation. Finally, a new algorithm, *geographically ranked guaranteed proportionality*, is proposed to circumvent the impossibility theorem.

1. Introduction

Electoral systems are fundamental to democracies, transforming votes into political power. It is essential for the legitimacy of electoral systems that the number of seats in parliament roughly corresponds to the number of votes for each party (proportionality) and different parts of the country get their local candidates represented (regional representation).

Two-tier electoral systems initially allocate seats within regional subdivisions, followed by a process which is often aimed at improving proportionality by compensating parties that were underrepresented regionally [18, 45]. As a result, such systems are often labeled proportional. For most systems, however, the regional seat allocation takes priority. In that case, there is no guarantee that a party's share of votes corresponds roughly to its share of seats in parliament. We say such a system is not guaranteed proportional.

Mathematical definitions of regionality and guaranteed proportionality enable the statement of an impossibility theorem: No electoral system satisfies regionality, guaranteed proportionality and a fixed parliament size. The proof mechanism is that the number of compensatory seats needed to guarantee proportionality increases with the number of parties. The impossibility theorem expands the tradition that includes Arrow's [1], Gibbard's [23] and Balinski-Young's [3] to complex electoral systems.

Due to the increased political fractionalization, the trade-off has surfaced in many real-world electoral systems this decade. In the Danish 2022 general election, a regional seat was not leveled out by the compensatory seats for the first time since 1947 [16]. This seat decided the majority in parliament, leading to a situation where, in traditional left-right terminology, the left bloc won despite receiving fewer votes than the right bloc [15] - a dramatic event in a two-tier system traditionally labeled as proportional.

Germany had an electoral system satisfying both regionality and guaranteed proportionality, and there the trade-off materialized in the growth of the Bundestag from the standard 598 seats to 736 in the 2021 federal election [12]. The debate sparked by these issues led to the 2023 electoral reform [11], where regionality is relaxed and the constituency winners are no longer ensured parliamentary representation. Similarly, Sweden, after experiencing slight disproportionality due to the insufficiency of leveling seats in 2010, debated and reformed its election law to abolish strict regionality [38]. In Norway, parties have won regional seats with very few votes [54, sec. 5.2.6.5], such as in 2005, when the party Venstre received 1 out of 5 seats in Finnmark, despite getting only 2.2% of the votes there [46].

The same mechanism is observed beyond two-tier compensatory systems: increased political fractionalization sharpens the trade-off between geographical and proportional representation. For instance, the United Kingdom's one-tier electoral system with first-past-the-post (FPTP) in each constituency serves as a stark example of disproportionality that can arise without leveling seats. In the 2024 general election, the Reform UK party received 14.3% of the national vote but secured only five seats of the total 650, while the Labour party won 411 seats with 33.7% of the vote [53], showcasing the extreme regionality of the system. According to The Economist [49] the 2024 election was the least proportional UK general election ever, measured by the Gallagher index [21].

1.1. **Omissions and clarifications.** Electoral systems in use around the world are very diverse. To build a simple mathematical framework encapsulating the important aspects of real-world systems, simplifying assumptions are necessary. We omit individual candidates and the large set of specialized local rules related to minority parties, race, religious groupings, disability, age or gender quotas etc. The mathematical framework used does not encompass preferential or transferable votes or ties. Electoral thresholds are also omitted from the main development and discussed separately in Section A.1.

We caution the reader that we consciously deviate from some of the traditional classifications in the literature [40]. This includes, but is not limited to, the distinction between single- and multi-member constituencies, which for our purposes are equivalent. For the aim of the paper, it is also unnecessary to distinguish between *parallel voting, coexistence, party-list proportional representation, mixed-member proportional representation*, and similar classifications, as they will be treated equally in mathematical terms. This approach may overlook certain granular distinctions, but it allows precise mathematical definitions of key concepts. While this mathematical perspective may obscure other valuable insights from political science (see e.g., [48, 9, 44, 26]), we hope that it nonetheless contributes to the ongoing discourse on electoral design and evaluation.

1.2. **Structure of the paper.** First, a mathematical framework for electoral systems, apportionment methods and one-tier systems is introduced. Building on those definitions in Section 3, the concepts of regionality and guaranteed proportionality are defined and used to prove the impossibility theorem. In Section 4, it is shown how each of the three sides of the trade-off is exhibited for real-world electoral systems. Section 5 concerns two- and multi-tier systems and the mathematical structure of such systems. Finally, in Section 6.1, a workaround and the relation of the theorem to double proportionality is discussed. See Table 3 for an overview of the notation and terminology introduced in the paper.

2. Defining electoral systems

2.1. **Election outcomes, seat distributions and electoral systems.** Define an election outcome as a matrix of votes cast. For convenience, label all electable entities as 'parties'. Throughout, let p denote the number of parties and c the number of constituencies. These constituencies can be either multi- or single-member; no distinction is made.

Definition 2.1 (Election outcome)**.** For positive integers c and p a (c, p)-*election outcome* is a $c \times p$ matrix V where the entries $V[i, j]$ are non-negative integers that should be interpreted as the votes that the jth party gets in the ith constituency. Let $\mathcal{V}_{c,p}$ be the set of all (c, p)-election outcomes, $\mathcal{V}_c$ be the set of all election outcomes with fixed c, and $\mathcal{V}$ be the set of all election outcomes.

The matrix V could easily be extended by an extra row for the blank and invalid votes as the $(c+1)$th constituency, but we omit this for the present discussion. Some countries, such as Russia, Germany, and Japan, employ *two-vote systems* where voters receive two ballots, one for local candidates and another for (national or regional) party lists. In those cases, the election outcome can be represented with two rows for each constituency, one for each ballot type.

Definition 2.2 (Seat distribution)**.** A (c, p)-*seat distribution* is a $c \times p$ matrix of non-negative integers S, where the matrix entry $S[i, j]$ is the number of seats allocated for the jth party from the ith constituency for $1 \leq i \leq c$ and $1 \leq j \leq p$. The set of all (c, p)-seat distributions we denote by $\mathcal{S}_{c,p}$, the set of all seat distributions for fixed c by $\mathcal{S}_c$ and let $\mathcal{S}$ denote the set of all seat distributions.

Electoral systems turn the votes in an election outcome into a seat distribution.

Definition 2.3 (Electoral system)**.** An *electoral system* $\texttt{ES}$ is a function $\texttt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$, where c and $\tilde{c}$ are positive integers.

That is, an electoral system takes an input election outcome $V \in \mathcal{V}_c$ defined in terms of c precincts (or constituencies) and outputs a seat distribution $S \in \mathcal{S}_{\tilde{c}}$ corresponding to seats in parliament for each party in $\tilde{c}$ constituencies.

2.2. **Amalgamations.** Within real-world electoral systems, electoral districts of course vary in significance. It rarely matters whether a vote is cast in one precinct or another. Amalgamations group such subdivisions.

Definition 2.4 (Amalgamation)**.** An *amalgamation* is a function $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ that regroups the c rows of $V \in \mathcal{V}_c$ into an *amalgamated election outcome* $AV \in \mathcal{V}_{\tilde{c}}$ with $\tilde{c}$ rows[1].

In the simple case, the amalgamation merges constituencies, but generally the new constituencies are allowed to "overlap", in which case the total vote count may not be preserved. For instance, the electoral systems of the Seychelles, Zimbabwe and Italy sum up the votes across all constituencies, which amounts to treating the entire country as an additional constituency with reserved seats (see Figure 1 for a graphical illustration). Amalgamations will also be instrumental in giving a rigorous definition of two-tier electoral systems in Section 5.1.

Example 2.5. Elections for *Lasanble Nasyonal* in the Seychelles consist of 26 (single-member) constituencies. However, for the 2020 general election, 44 separate polling stations were established[2] [43]. Additionally, the Seychelles employs a system where party votes are also aggregated at the national level and additional seats are awarded. Since four parties

[1]Mathematically, an amalgamation is a $\tilde{c} \times c$ matrix with entries 0 and 1, acting on V by left-multiplication. The entries $A[r, l] = 1$ if and only if the lth row of V is summed into rth row of AV.

[2]For simplicity, we disregard the fact that some of these polling stations allowed nursing home residents, hospital patients, and voters on outlying islands to cast their votes for a constituency other than the one hosting the station.

competed in the election [14], it is natural to use the following amalgamation to get a 27×4 matrix (see also Figure 1).

$$A \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} \\ v_{2,1} & v_{2,2} & v_{2,3} & v_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ v_{43,1} & v_{43,2} & v_{43,3} & v_{43,4} \\ v_{44,1} & v_{44,2} & v_{44,3} & v_{44,4} \end{bmatrix} = \begin{bmatrix} v_{1,1}+v_{2,1} & v_{1,2}+v_{2,2} & v_{1,3}+v_{2,3} & v_{1,4}+v_{2,4} \\ v_{3,1} & v_{3,2} & v_{3,3} & v_{3,4} \\ \vdots & \vdots & \vdots & \vdots \\ v_{42,1}+v_{43,1} & v_{42,2}+v_{43,2} & v_{42,3}+v_{43,3} & v_{42,4}+v_{43,4} \\ v_{44,1} & v_{44,2} & v_{44,3} & v_{44,4} \\ \sum_i v_{i,1} & \sum_i v_{i,2} & \sum_i v_{i,3} & \sum_i v_{i,4} \end{bmatrix}.$$

Here the 44 original rows are summed into 26 new rows (one per constituency), and a 27th row with the national vote totals is appended. The electoral system is discussed in Example 2.11ii).

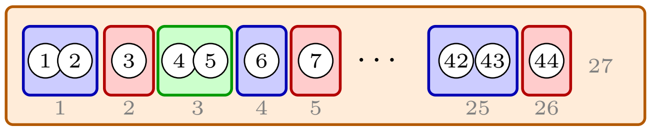


FIGURE 1. Illustration of the concept of amalgamation. The amalgamation $A : \mathcal{V}_{44} \to \mathcal{V}_{27}$ groups 44 original constituencies (numbered circles) into 27 amalgamated constituencies (colored boxes), where the 27th row represents the national total (the encompassing orange box).

2.3. **Apportionment methods.**

Apportionment methods determine how votes are converted into seats within individual constituencies and serve as building blocks for more complex electoral systems.

Definition 2.6 (Apportionment method)**.** An *apportionment method* is a function $\mathtt{M} : \mathcal{V}_{1,p} \to \mathcal{S}_{1,p}$ that for every tuple of votes $v = (v_1, \dots, v_p)$ outputs a tuple of seats $\mathtt{M}(v) = (m_1, \dots m_p)$.

An apportionment method $\mathtt{M}$ satisfies *party invariance* (also called *neutrality*) if the outcome of the apportionment method does not depend on the ordering of the parties. It satisfies *monotonicity* (cf. Concordance [36, Sec. 4.3]) if party 1 receives more votes than party 2 ($v_1 > v_2$) then the number of seats for party 1 is greater or equal to the number of seats for party 2 ($m_1 \geq m_2$). Notice that monotonicity implies that the largest party always gets at least one seat. We refer to that property as the *winner-take-one rule*. If there is a fixed number k of seats available in a given constituency independently of votes cast v, we say that the method has *fixed seat count* and use the shorthand notation $\mathtt{M}^k$ for the apportionment method $\mathtt{M}$ distributing k seats. In the following, it is always assumed that the apportionment methods (but not general electoral systems) are party invariant and monotone.

Common examples of apportionment methods that satisfy all these assumptions are the D'Hondt method (DH), "first-past-the-post" (FPTP), Hare-LR (LR) (the method of largest remainders[3]) and the Sainte-Laguë method (SL). Standard apportionment methods typically fall into two classes, divisor methods and quota methods, and one way to phrase the Balinski-Young theorem is that these classes are irreconcilable [3]. For a general definition of the methods in these classes see e.g., [36, 28]. Let us give some not so standard examples of apportionment methods that illustrate notation and the diversity of electoral systems. The examples are not essential for understanding the rest of the paper.

Example 2.7 (Apportionment methods)**.**

(i) (United States *Electoral College* for presidential elections) First-past-the-post is mostly used for single-member constituencies, with the US Electoral College as an important exception. Here all US states except Maine and Nebraska employ FPTP with $k_i > 1$. The algorithm is simple. Suppose $v = (v_1, \dots, v_p)$ is *size-ordered* (such that $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_p$), then
$$\mathtt{FPTP}^k(v) = (k, 0, \dots, 0).$$

(ii) (Cameroon's lower house of parliament *L'Assemblée Nationale*) All of Cameroon's 58 single- and multi-member constituencies use the following apportionment method: Any list that gains a majority of the vote wins all of the seats. Otherwise, the list in first place receives half of the seats, rounded up to the nearest whole number. The remaining seats are then distributed among the other parties above the threshold using Hare-LR (LR). For $v = (v_1, \dots, v_p)$ size-ordered, then (with $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ denoting rounding up and down)
$$\mathtt{M}^k_{\text{Cameroon}}(v) = \begin{cases} (k, 0, \dots, 0) & \text{if } v_1 > 0.5 \sum_j v_j \\ \left(\lceil \frac{k}{2} \rceil, \mathtt{LR}^{\lfloor \frac{k}{2} \rfloor}(v_2, \dots, v_p) \right) & \text{else.} \end{cases}$$

[3]Hare quota with largest remainders $\mathtt{LR}^k$: Define $q_j = k\, v_j / \sum_u v_u$, allocate $\lfloor q_j \rfloor$, then give the remaining seats to the parties with the largest remainders $r_j = q_j - \lfloor q_j \rfloor$.

(iii) (Armenia's unicameral parliament Ազգային ժողով (*Azgayin Zhoghov*) initial phase[4] [39]) The electoral system of Armenia[5] has a single constituency. Initially, 101 seats are distributed using a modified variant of the largest remainders method where the remainders of the largest parties are rounded up. Afterwards, the seat distribution is amended: If a party secures a majority of the seats but not 54%, it is allocated additional seats until it reaches 54%. If any party wins over 67% of the seats, the remaining parties are given extra seats so that the largest party holds at most 67%. Note that this method does not have fixed seat count.

2.4. **One-tier electoral systems.** The impossibility theorem that is the topic of the next section applies to all electoral systems, but it is mainly non-trivial for compensatory electoral systems with multiple tiers. When regionality of an electoral system is defined in the next section, it will refer to the system's one-tier basis of regionally apportioned seats (and whether parties can lose such seats in further calculations).

In practice, most electoral systems do not have interacting upper tiers and will hence be classified as one-tier. The definition aims both at formally writing down the structure of one-tier systems and it is crucial input for the definition of regionality (cf. Definition 3.2). For a matrix M, let $M[i,\cdot]$ denote the ith row and $M[\cdot,j]$ the jth column.

Definition 2.8 (One-tier electoral system)**.** An electoral system $\mathtt{ES}_{\text{1-Tier}} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ is *one-tier* if there exists an amalgamation $A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$ and a tuple of apportionment methods $\mathtt{M} = (\mathtt{M}_1, \dots, \mathtt{M}_{\tilde{c}})$ such that

$$\mathtt{ES}_{\text{1-Tier}}(V)[i,\cdot] = \mathtt{M}_i((AV)[i,\cdot]). \tag{1}$$

In other words, one-tier systems are those in which the resulting seat distribution can be determined by applying the apportionment method independently to the votes for each amalgamated constituency $\tilde{c}$, without any interaction between them, see Figure 2 for an illustration. The following example is illustrative.

Example 2.9 (The Latvian *Saeima*)**.** The country is split into 5 multi-member constituencies electing 12, 13, 13, 26 and 36 members using the Sainte-Laguë method (SL). In the 2022 election, 1001 polling stations were used [33]. Thus, the system is defined by the amalgamation $A : \mathcal{V}_{1001} \to \mathcal{V}_5$ and the tuple of apportionment methods $(\mathtt{SL}^{12}, \mathtt{SL}^{13}, \mathtt{SL}^{13}, \mathtt{SL}^{26}, \mathtt{SL}^{36})$. Here we ignore the electoral threshold (5% in the Latvian case).

One-tier systems are common throughout the world, cf. Figure 5, especially among former British territories where FPTP is often used.

Example 2.10 (Trivial One-tier systems)**.**

(i) (The United Kingdom's *House of Commons*) First-past-the-post voting is used in all 650 single-member constituencies (as of 2024), which corresponds for every $1 \le i \le 650$ to the apportionment method $\mathtt{M}_i = \mathtt{FPTP}^1$.

(ii) (The Netherlands' *Tweede Kamer der Staten-Generaal*) The Dutch electoral system uses the D'Hondt method to distribute 150 seats in a single constituency. Thus, it is a one-tier system with apportionment method $(\mathtt{DH}^{150})$.

A one-tier system where the methods $\mathtt{M}_i$ are all of the same type is called *homogeneous*, and if they are different the one-tier system is called *parallel* [40]. The following examples show how versatile Definition 2.8 is. The next example shows how many mixed member majoritarian systems become one-tier under the definition. This formalizes the part of the literature that claims that parallel national allocations should not be called tiers [45, p.59].

Example 2.11 (Complicated One-tier systems)**.**

(i) (Russia's *State Duma*) Each voter is given two votes. The first vote is used to elect a local candidate in one of the 225 single-member constituencies using the first-past-the-post method. The second vote is cast for a party list and is tallied nationally. This way another 225 seats are distributed using the Hare quota rule (with a 5% electoral threshold) $\mathtt{LR}_{5\%}$. The corresponding tuple of apportionment methods is $(\mathtt{M}_1, \dots, \mathtt{M}_{226}) = (\mathtt{FPTP}^1, \dots \mathtt{FPTP}^1, \mathtt{LR}^{225}_{5\%})$. Note that defining this system as one-tier deviates somewhat from the literature.

(ii) (The Seychelles *Lasanble Nasyonal*) The Seychellois electoral system distributes 26 seats using first-past-the-post in single-member constituencies. In addition, up to 10 additional seats are allocated nationally - one per 10% of the total national vote received by a party, that is, $\mathtt{M}_{\text{Seychellois Bonus}}(v)[j] = \lfloor \frac{10 v_j}{\sum_j v_j} \rfloor$. So with the amalgamation in Example 2.5, the electoral system is one-tier, with $(\mathtt{M}_1, \dots, \mathtt{M}_{27}) = (\mathtt{FPTP}^1, \dots \mathtt{FPTP}^1, \mathtt{M}_{\text{Seychellois Bonus}})$.

(iii) (The United States *Electoral College*) for US presidential elections might at first sight look parallel due to Maine and Nebraska not using FPTP (cf. Example 2.7). In the two states, the largest party in each congressional district is awarded a seat and the statewide largest party is awarded two additional seats. So, upon amalgamation, the method in each state is $(\mathtt{FPTP}^1, \dots, \mathtt{FPTP}^1, \mathtt{FPTP}^2)$. As a consequence, the entire Electoral College is homogeneous.

[4] If no party (or a coalition of parties formed within six days after the election) can obtain a majority, a second round of elections will be held in which the two best-performing political forces will participate. All seats received in the first round will be retained. The party (or newly formed coalition) that wins the second round will be allocated additional seats to reach 54% of all seats.

[5] In this example, the electoral threshold and the reserved seats for Assyrians, Kurds, Russians, and Yazidis are disregarded.

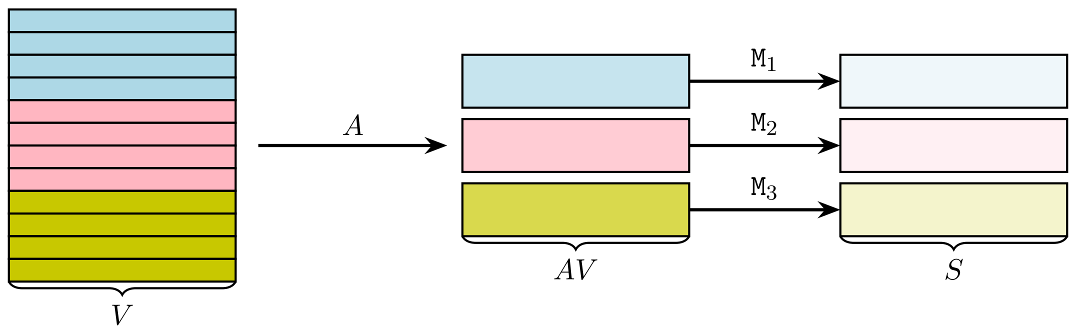


FIGURE 2. Graphical representation of the definition of one-tier systems Definition 2.8. In this case the amalgamation A sums up rows and afterwards the one-tier electoral system acts row-wise.

The mathematical definition of one-tier systems enables a strict classification. One consequence of this classification is that run-off systems are not classified as one-tier (or two-tier for that matter), as such systems take two election outcomes as input. The following examples are borderline cases that are not quite one-tier systems.

Example 2.12 (Almost but not quite one-tier systems)**.**

(i) (The *European Parliament*) electoral system consists of separate national elections held by each of the 27 member states, without interaction between them and with different methods $\mathtt{M}$ which, in the cases of Italy and Poland, are themselves not one-tier systems and therefore not quite apportionment methods, see e.g., [37].

(ii) (The United States *House of Representatives*) uses $\mathtt{FPTP}$ across the constituencies in 46 states, but ranked-choice or runoff systems are employed in Alaska, Maine, Georgia and Louisiana, so it is not one-tier.

3. Impossibility theorem

To state the impossibility theorem, regionality, proportionality and fixed parliament size are defined. The definitions are primarily motivated by two- and multi-tier systems (to be defined in Section 5.1) that often aim to balance regionality and proportionality. Fixed parliament size contributes to simplicity and helps control costs[6].

3.1. **Properties of electoral systems.**

The simplest property is fixed parliament size.

Definition 3.1 (Fixed parliament size)**.** An electoral system $\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ has *fixed parliament size* if the total number of distributed seats is constant for all $V \in \mathcal{V}_c$. That is, there exists a positive integer m such that for any number of parties p and any $V \in \mathcal{V}_{c,p}$, $\sum_{i=1}^{\tilde{c}} \sum_{j=1}^{p} \mathtt{ES}(V)[i,j] = m$.

In practice, most electoral systems have fixed parliament size. Notable exceptions[7] are the parliament of New Zealand *Premata Aotearoa* and the one-tier system of the Seychelles (Example 2.11ii).

Regionality reflects whether parties always keep the *regional seats* they initially won in the constituencies with $\mathtt{ES}_{\text{1-Tier}}$.

Definition 3.2 (Regionality)**.** An electoral system $\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ *respects regionality* with respect to a one-tier electoral system $\mathtt{ES}_{\text{1-Tier}} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ if for all $V \in \mathcal{V}_c$, and each i, j, it holds that $\mathtt{ES}_{\text{1-Tier}}(V)[i,j] \leq \mathtt{ES}(V)[i,j]$.

For electoral systems satisfying regionality say the regional seats are *final*. Such regional seats are sometimes perceived differently than the compensatory seats [34]. Most two and multi-tier electoral systems respect regionality, cf. Table 1 in Section 4. In fact, $\mathtt{ES}$ is usually[8] built from an $\mathtt{ES}_{\text{1-Tier}}$ with an additional way of distributing compensatory seats.

Following Balinski-Young [3, Section 2], say that an apportionment method $\mathtt{M}^k$ is *weakly proportional* if it divides proportionally, whenever possible. That is, when there are no fractional parts in the largest remainders method. Formally, if $(v_1, \ldots, v_p)$ is a vector of votes and v is their sum, $\mathtt{M}^k$ is weakly proportional if $\mathtt{M}^k(v_1, \ldots, v_p) = (\frac{kv_1}{v}, \ldots, \frac{kv_p}{v})$ whenever $\frac{kv_j}{v}$ is an integer for each j. All the (traditional) apportionment methods Hare-LR ($\mathtt{LR}$) (Hamilton), D'Hondt ($\mathtt{DH}$) (Jefferson), Lowndes, Sainte-Laguë ($\mathtt{SL}$) (Webster), Adams, Dean and Hill are weakly proportional [3, Proposition 2.1 in Appendix A].

[6]Additionally, maintaining a fixed and odd number of seats can help prevent legislative deadlocks by ensuring a decisive majority in votes, further strengthening the stability and functionality of the system.

[7]There are other exceptions. In some cases, seats may be left vacant when the list of candidates is insufficient to accommodate the allocated seats, which is the case for South Korea's National Assembly considered in [30, article 189 (5)]. A curious example occurred in the 2021 Kyrgyz parliamentary election, where two single-member constituency seats remained unfilled because the *against all* option received the most votes. While such cases exist, they fall outside the scope of our discussion.

[8]Additionally, note that the electoral system $\mathtt{ES}_{\text{1-Tier}}$ is not completely general, as it assumes that all of the apportionment methods $\mathtt{M}^{k_i}$ have a fixed seat count. Two-tier systems that do not satisfy this requirement include the current Polish subelectoral system in the European parliamentary elections [37] and the system employed at Romanian parliamentary elections from 2008-2012 [24].

Say that an electoral system $\texttt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ *agrees with* $\texttt{M}$ if for all $V \in \mathcal{V}_c$,

$$A_{\text{total}}\texttt{ES}(V) = \texttt{M}(A_{\text{total}}V), \tag{2}$$

where A_{total} is a total amalgamation. In other words, when the distributed seats $\texttt{ES}(V)$ are summed up the electoral system matches the function of the apportionment method $\texttt{M}$.

Definition 3.3 (Guaranteed proportionality)**.** An electoral system $\texttt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ is *guaranteed proportional* if there exists a weakly proportional apportionment method $\texttt{M}$ as defined in Definition 2.6 such that $\texttt{ES}$ agrees with $\texttt{M}$ for all elections V.

For two-vote systems, a bit of flexibility should be allowed and guaranteed proportionality should be defined only with respect to one vote type (row in an amalgamation).

Guaranteed proportionality is a structural property. Many well-functioning two-tier systems traditionally labeled as *proportional systems*, are not guaranteed proportional even if all previous elections have been consistent with the proportional method used.

3.2. **Impossibility theorem.** A mismatch between the number of seats and votes in each constituency relatively easily causes problems with having regionality, proportionality and fixed parliament size[9].

This does indeed seem to be a real-world issue. Examples of countries where the number of seats per voter varies significantly across constituencies include Norway and Denmark, with Finnmark and Bornholm respectively being over-represented relative to what their sparse populations would suggest. For current operational two-tier systems, k_i is often determined in advance based on various methods[10]. To avoid this mismatch, the strongest possible restriction is to insist that the number of regional seats $(k_1, \ldots, k_c)$ is distributed weakly proportionally according to the sum of the votes across the constituencies. Call such election outcomes *vote-seat consistent*. Even in this case, the impossibility arises:

Theorem 3.4 (Impossibility theorem)**.** For any electoral system $\texttt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ and any one-tier electoral system $\texttt{ES}_{\text{1-Tier}} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ with apportionment methods $(\texttt{M}_i^{k_i})_{1 \le i \le \tilde{c}}$, having $k_i \geq 1$ mandates for at least two distinct indices i, there exists a vote-seat consistent election outcome where at least one of the following desiderata fails:

(i) Regionality: $\texttt{ES}$ respects regionality with respect to $\texttt{ES}_{\text{1-Tier}}$.
(ii) Fixed parliament size: The system $\texttt{ES}$ has fixed parliament size.
(iii) Proportionality: $\texttt{ES}$ is guaranteed proportional.

Proof. Assume that all three desiderata are satisfied and let m denote the fixed parliament size. We construct a vote-seat consistent election outcome V that leads to an impossibility whenever

$$p > \frac{T}{C-1} + \max(k_1, \ldots, k_C) + 1, \tag{3}$$

where p is the total number of parties, $T = m - \sum_{i=1}^{C} k_i$ is the number of non-regional seats, and C is the number of constituencies with $k_i \geq 1$.

For the construction, suppose there are $p_r = \max(k_1, \ldots, k_C) + 1$ red parties $P_1, \ldots, P_{p_r}$ and p_b blue parties $N_1, \ldots, N_{p_b}$. In constituency i, each blue party gets $k_i a$ votes where a is to be specified later. The first red party P_1 receives $k_i(a+2)$ votes in each constituency $1 \le i \le C-1$ and 0 votes when $i = C$. The parties $P_2, \ldots, P_{p_r - 1}$ get $k_i(a+1)$ votes for each i. The last party P_{p_r} gets 0 in every constituency except the last one, where it gets $k_C(a+2)$. If $k_i = 0$ for some constituencies we assume that no votes were cast in those constituencies (this is still vote-seat consistent). The matrix form of the election outcome is

$$V = \left[\begin{array}{ccccc|ccccc}
\multicolumn{5}{c|}{\overbrace{\hphantom{k_{C-1}(a+2)\quad k_{C-1}(a+1)\quad\cdots\quad k_{C-1}(a+1)\quad k_C(a+2)}}^{p_r \text{ red parties}}} & \multicolumn{4}{c}{\overbrace{\hphantom{k_{C-1}a\quad k_{C-1}a\quad\cdots\quad k_{C-1}a\quad k_{C-1}a}}^{p_b \text{ blue parties}}} \\
k_1(a+2) & k_1(a+1) & \cdots & k_1(a+1) & 0 & k_1 a & k_1 a & \cdots & k_1 a & k_1 a \\
k_2(a+2) & k_2(a+1) & \cdots & k_2(a+1) & 0 & k_2 a & k_2 a & \cdots & k_2 a & k_2 a \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
k_{C-1}(a+2) & k_{C-1}(a+1) & \cdots & k_{C-1}(a+1) & 0 & k_{C-1} a & k_{C-1} a & \cdots & k_{C-1} a & k_{C-1} a \\
0 & k_C(a+1) & \cdots & k_C(a+1) & k_C(a+2) & k_C a & k_C a & \cdots & k_C a & k_C a
\end{array}\right].$$

In the ith constituency, the total number of votes is $v_i = k_i\Big((p_r - 1)(a+1) + 1 + a p_b\Big)$, so V is vote-seat consistent.

With the vote distribution described, the contradiction is derived from the assumption that all three desiderata are satisfied.

[9]Indeed, even with only two parties and three constituencies: If one party is the largest party in two constituencies they get at least 2 seats with a total of v_1 votes, but by having many votes in the last constituency the second party may in total get $(m-1)v_1$ votes, where m is the fixed parliament size. By weak proportionality, the mandates should be distributed $(1, m-1)$.

[10]Examples currently used include the number of votes cast in the constituency in the last election, the population in the constituency at the last census, the surface area of the constituency, and politically negotiated minimum number of representatives.

Since the first red party P_1 is the largest party in each of the first $C-1$ constituencies, by winner-take-one for each of the methods $\mathtt{M}_i$ it will win at least $C-1$ regional seats, with a total of $(a+2)\sum_{1\leq i\leq C-1} k_i$ votes. Since ES satisfies regionality with respect to $\mathtt{ES}_{\text{1-Tier}}$ defined by $(\mathtt{M}_i)_{1\leq i\leq \tilde{c}}$, the party P_1 obtains at least $C-1$ seats in $\mathtt{ES}(V)$.

Now consider the blue parties. In each constituency i, none of the blue parties $N_1, \ldots, N_{p_b}$ are among the k_i largest parties, so by monotonicity of the method $\mathtt{M}_i$, each of them receives 0 regional seats. However, each blue party gets in total $a\sum_{1\leq i\leq C} k_i$ votes. Choosing the free parameter a such that $a > \sum_{1\leq i\leq C-1} \frac{2k_i}{k_C}$ then each blue party has strictly more total votes than P_1. By monotonicity of the apportionment method that ES is guaranteed proportional with respect to, each blue party must then receive at least $C-1$ seats in total.

Now, since we assume fixed parliament size m, also the number of non-regional seats T is fixed. We have just argued that each of the p_b blue parties gets at least $C-1$ of these non-regional seats. Hence $T \geq (C-1)p_b$. For sufficiently large p_b, this contradicts the fixed parliament size constraint, since T is fixed and cannot grow with p_b.

Therefore, it is impossible to satisfy all three desiderata simultaneously. The bound (3) follows since $p = p_r + p_b$. □

At this point, a couple of remarks are in order.

Remark 3.5 (Numerical example)**.** For electoral systems that have first-past-the-post in the lower tier ($k_i = 1$) with 100 constituencies and 10 parties, the impossibility arises whenever the number of leveling seats T is less than $(10-2)\cdot 99 = 792$. Thus, the construction applies to realistic cases.

Remark 3.6 (One-tier systems and the Netherlands)**.** The theorem also applies to one-tier systems, which usually satisfy regionality and fixed parliament size, but not guaranteed proportionality. In a sense, the trivial Dutch electoral system described in Example 2.10ii) satisfies all three properties, but there is only a single constituency, so it is not true that $k_i \geq 1$ for at least two i.

Remark 3.7 (Many parties complicate proportionality and regionality)**.** The proof hinges on the possibility of many (small) parties, which is one concrete mechanism exposing the underlying tension. More generally, impossibility can arise in several different ways: distorted districts, many parties, the apportionment methods employed, and combinations thereof. The mechanism of many parties has a suggestive parallel with the recent increased number of parties in European politics, with four parties achieving more than 10% in the UK, with the CSU and Die Linke making it to the Bundestag in Germany in 2021 only on direct seats [12]. Similarly, the Danish parliamentary election in 2022 saw a record-breaking number of parties entering parliament [15]. This suggests that the proof elucidates the mathematical mechanism behind these developments. While these examples are recent, the effective number of parties has been increasing for many years, increasing the relevance of the trade-off exhibited in Theorem 3.4.

Remark 3.8 (Micromega rule)**.** The bound (3) can be viewed as an incarnation of the micromega rule, which describes the effect whereby large parties benefit from smaller constituencies (they obtain over-representation without adjustments). For a fixed total number of regional seats, if the constituencies become smaller, the number of constituencies increases and the number of parties needed to construct the counterexample decreases.

Remark 3.9 (Relation to the impossibility result of Demange [10])**.** In [10], Demange considered the related problem of allocating seats across both parties and constituencies (bi-allocation). Demange introduced the concept of district decentralization, meaning that the seat distribution in each constituency depends solely on the votes cast in that constituency. District decentralization is a much stronger requirement than regionality, which only requires some of the seats to be determined locally. The impossibility result [10, Proposition 2] in the setup of real numbers is that if the district allocation is distorted (not vote-seat consistent over the reals) then any bi-allocation method cannot be both district decentralized and party proportional (perfectly proportional over the reals). Demange further illustrates through the German and Italian examples that introducing integer requirements cannot solve these problems. From one perspective, Theorem 3.4 is a formalized stronger version of this statement: over the integers, no party-proportional method that distributes even some of the seats district decentralized exists, regardless of whether district seat totals match district vote totals. See Section 6 for further discussion.

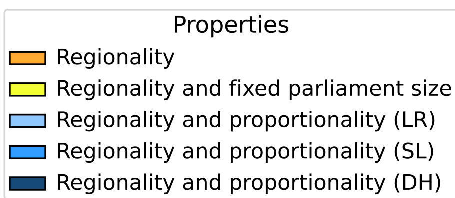
Remark 3.10 (Bounding the number of required leveling seats in terms of the number of parties)**.** The impossibility theorem refines the discussion in the literature about the number of compensatory seats necessary to ensure proportional representation [44]. The theorem gives an answer, namely that if there is no bound to the number of parties, then there is no definite number. A natural next mathematical question is whether the bound (3) on the number of parties p can be tightened by constructing more sophisticated election outcomes. Additionally, one can attempt to construct bounds that link other quantities, such as T and $\sum k_i$. This direction for further work could also involve assumptions about the relation between the number of votes and seats in each constituency. In addition, the mathematically attentive reader might notice that only the monotonicity of ES was used and the full power of weak proportionality was not needed. Better bounds can be constructed by assuming that ES is guaranteed proportional with respect to, for example, SL or LR.

4. Real-world examples

The impossibility theorem gives a formal perspective on real-world examples and present discussions (cf. Section 1). In Table 1, a classification of multi-tier systems shows how all sides of the trade-off have been explored. See also Figure 3 that shows how regional German parliaments exhibit a variety of combinations.

Country	**Regionality**	**Guaranteed proportionality**	**Fixed parliament size**
Germany_{2021}	✓	✓	×
Sweden, Germany_{2025}, $\text{Denmark}_{1948-1953}$	×	✓	✓
Austria, Bolivia, Bosnia-Herzegovina, Denmark, Dominican Republic, Estonia, Iceland, Lesotho, Norway, Scotland Slovenia, South Africa, $\text{South Korea}_{2020}$	✓	×	✓
New Zealand, Singapore	✓	×	×
$\text{Romania}_{2008-2012}$	×	✓	×

Table 1. Broad overview of the properties of some multi-tier electoral systems for national parliaments, excluding the electoral threshold (discussed in Section A.1) and other specialized rules, the table does not capture all system details[16]. Only multi-tier systems with a compensatory element are listed (other non-compensatory multi-tier systems such as Hungary and Mexico are left out[17]). Here the two German systems are guaranteed proportional with respect to Sainte-Laguë/Schepers method, the Romanian to D'Hondt method and the Swedish to a modified Sainte-Laguë method.



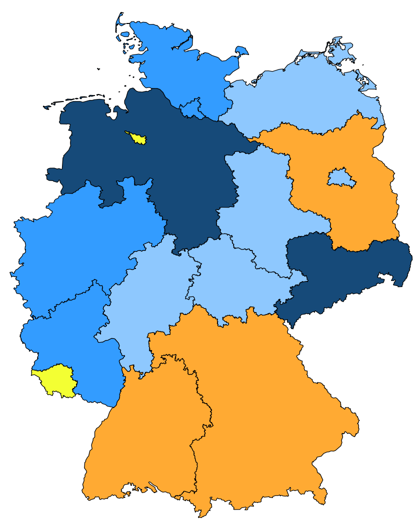


Figure 3. Overview of the electoral systems used for regional German parliaments (Ger. *Landtage*) and the apportionment methods that some are guaranteed proportional with respect to. In contrast to the new federal German system, they all satisfy regionality. Data source: [31]

To decode Table 1, it is instructive to study the Icelandic system, which is arguably the simplest two-tier system in use. Recall that two-tier systems, informally speaking, first distribute regional seats (with a one-tier basis) followed by the distribution of additional (often compensatory) seats in an upper tier. Building on the following example, a precise definition of two-tier electoral systems is given in Section 5.

Example 4.1 (Iceland's unicameral parliament *Alþingi*)**.** The Icelandic parliament comprises 63 members: 54 elected directly from six multi-member constituencies and the remaining 9 through compensatory seats. The first tier has six constituencies distributing 7, 9, 9, 9, 9 and 11 regional seats [50, Art. 8] using the D'Hondt method (illustrated in Figure 4). Thus, the one-tier basis of the system can be written as $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}} = (\mathtt{DH}^7, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^{11})$. This one-tier system is the natural system that regionality in Theorem 3.4 should be defined with respect to. All entries in Table 1 have a similar one-tier system as its basis.

[11] Germany_{2025} does not have fixed parliament size if one party gets more than 50% of the vote in which case extra seats are added to ensure that they obtain at least 50% of the seats. In Slovenia two additional deputies are elected by the Italian and Hungarian minorities.

[12] Singapore is classified as two-tier due to the *non-constituency member of parliament system*, where up to 12 non-winning candidates are elected and enjoy all of the privileges of ordinary members of parliament, apart from a substantially lower salary [8].

Now, the compensatory seats are distributed nationally in the following way[13] [50, Art.108]: The national list of D'Hondt quotients is generated[14] and ordered by size. The 54 regionally distributed seats correspond to 54 entries on this list. Mark these entries as final. The remaining 9 seats are distributed to the parties corresponding to the 9 largest quotients that are not already marked. See Figure 4 for a graphical illustration of this procedure for the Icelandic election of 2024.

If all the 54 initially marked quotients are among the 63 largest, the national seat totals will be distributed according to the D'Hondt method. For such election outcomes, equation (2) in Definition 3.3 holds true. But if one of the 54 initially marked quotients is number 64 or further down the list, the national total is not distributed corresponding to the D'Hondt method on the national totals. Thus, the Icelandic electoral system is not guaranteed proportional, but it does satisfy regionality and fixed parliament size.

#	Party	Quotient	Outcome
1	**S** *Samfylkingin*	44,091	*constituency seat*
⋮		*(ranks 2 – 43)*	
44	**B** *Framsókn*	4,145	**compensatory seat**
⋮		*(ranks 45 – 50)*	
51	**F** *Flokkur f.*	3,661	**compensatory seat**
52	**D** *Sjálfst.*	3,429	*constituency seat*
53	**S** *Samfylkingin*	3,392	*constituency seat*
54	**C** *Viðreisn*	3,361	*constituency seat*
55	**B** *Framsókn*	3,316	**compensatory seat**
56	**F** *Flokkur f.*	3,254	**compensatory seat**
57	**M** *Miðfl.*	3,212	*constituency seat*
58	**D** *Sjálfst.*	3,165	**compensatory seat**
59	**S** *Samfylkingin*	3,149	*constituency seat*
60	**C** *Viðreisn*	3,055	**compensatory seat**
61	**S** *Samfylkingin*	2,939	**compensatory seat**
62	**D** *Sjálfst.*	2,939	**compensatory seat**
63	**F** *Flokkur f.*	2,929	**compensatory seat** *63 seats total*
64	**M** *Miðfl.*	2,855	not awarded
65	**C** *Viðreisn*	2,800	not awarded

Compensatory Constituency Not awarded

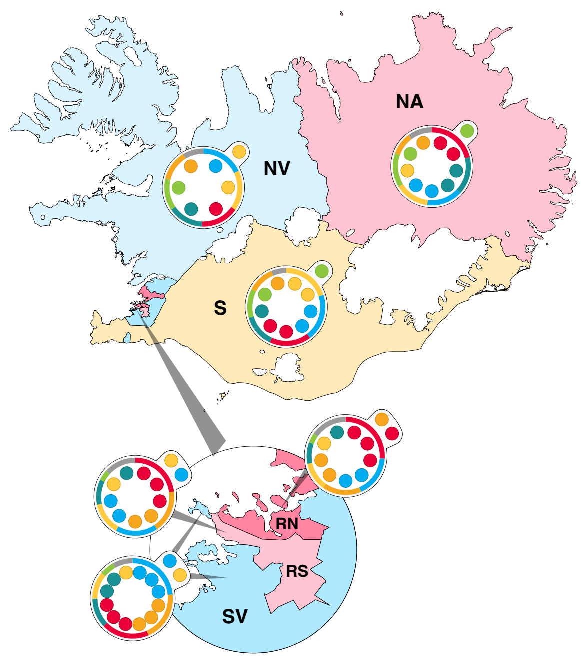


Figure 4. The 2024 Icelandic election. Left: Distribution of compensatory seats by D'Hondt quotient ranking. Compensatory seats have bold outlines; constituency seats are faded; hatched rows were not awarded. Right: The six multi-member constituencies, where colored inner circles show regional seats and outer circles show compensatory seats. Only parties above the 5% national threshold are shown. Data sources: [32, 5].

As in the Icelandic case, any two-tier system striving for proportionality and satisfying regionality must have some mathematical mechanism for the case when one or more parties obtain more regional seats than can be leveled out using additional tiers. The redistribution comes in two flavors, if the regional seats are final the national seats must be redistributed and if the system is guaranteed proportional some regional seats must be "taken back".

4.1. **Regional seat redistribution under guaranteed proportionality and fixed parliament size.** The 2023 reforms to the German electoral law are summarized in Table 1. Previously, the parliament size was not fixed. The increasing number of parties winning representation in the Bundestag led to an increase in the number of additional regional seats that some parties obtained without appropriate nationwide votes to back them up (Ger. *Überhangmandate*, Eng. *Overhang* seats) and the seats that other parties were additionally awarded to compensate for that (Ger. *Ausgleichsmandate*). Consequently, the German parliament expanded in size from the standard 598 to 736.

[13]We only consider the national distribution and not how they are distributed to the constituencies.

[14]That is for every party the total number of votes is divided by 1, 2, 3,.. etc.

In the German system introduced in 2023 (see [7]), some overhang seats (that is seats without *Zweitstimmendeckung* in new German terminology) are given up. The algorithm for choosing which seats to give up is complicated and depends on the coarser structure of the Bundesländer (see for example the fictitious use of the law on the 2021 election [13]). This complication makes the system multi-tier in contrast to the former German system, which was two-tier. It is a central feature of the algorithm for retracting regional seats that it prioritizes the seats by the percentage of the votes the locally elected candidate obtained.

Similarly, for the electoral system of the Swedish parliament (*Riksdagen*), regionally distributed mandates can be "taken back" to ensure national proportionality. As a consequence, the system does not satisfy regionality. Here, the overhang seats are redistributed locally in each constituency one by one, starting with the seat corresponding to the least quotient [47, Kap. 14]. In Section 6.1, yet another redistribution algorithm is devised.

4.2. **National seat redistribution under regionality and fixed parliament size.** When regional seats are final and the system aims at but does not achieve guaranteed proportionality, the system needs a mechanism for redistributing national seats. The natural choice depends on whether the system aims at guaranteed proportionality with respect to a divisor or a quota method.

Divisor methods: In the Icelandic case (Example 4.1), the national seats are ordered by the size of their D'Hondt divisor. Naturally, the national seats that are left out are the ones with the lowest quotients.

Quota methods: Systems allocating national seats based on Hare quota do not have a natural ordering of the quotients[15]. Therefore, the electoral systems of many countries, including Denmark, Norway, South Africa, and Bosnia-Herzegovina, are built on an iterative procedure where new Hare quotas are calculated omitting overrepresented parties. A disadvantage of iterative solutions is their increased complexity. This can become so overwhelming that maintaining a clear overview is difficult. Since electoral systems are typically defined using natural language in legal texts, it is possible that there exist election outcomes for which the algorithm is ambiguous or not well-defined [35, 41]. This issue was demonstrated for the Danish electoral system in [17], and we suspect that similar ambiguities could exist in other electoral systems as well.

In New Zealand, overhang seats are not redistributed and thus regionality is satisfied. Parties still obtain the number of seats corresponding to their vote share with respect to the initial size of the parliament (using the Sainte-Laguë method) - potentially leading to an increasing size of the parliament (two seats in the 2023 election). Since parties without overhang seats are not compensated, the system is not guaranteed proportional.

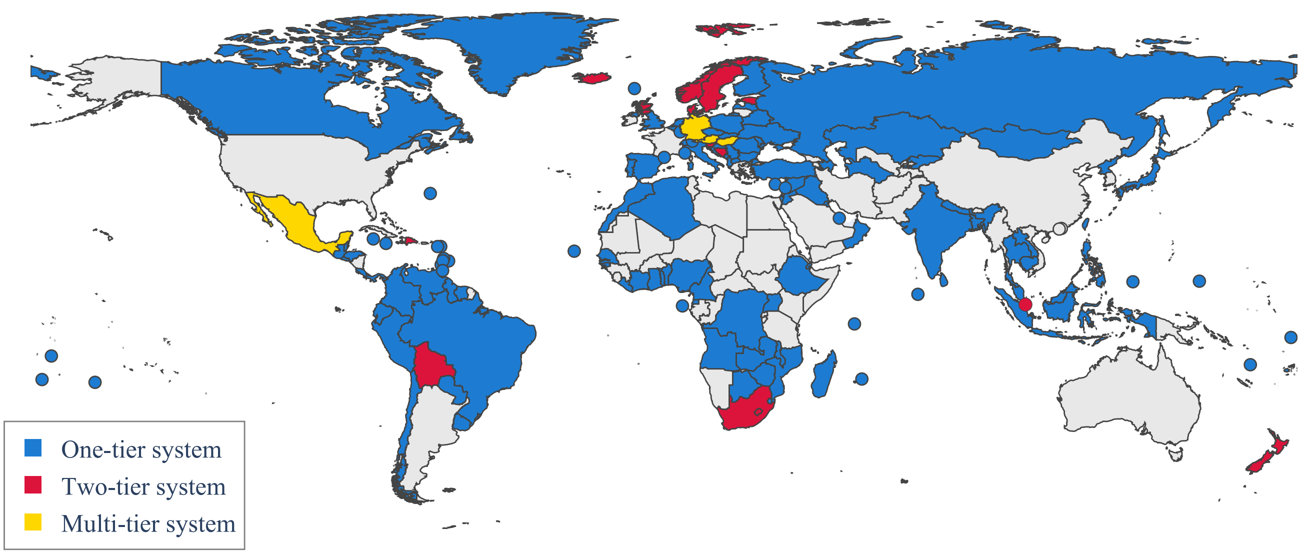


Figure 5. An overview of countries that as of 2024 employ one-tier, two-tier or multi-tier systems for elections to the lower (or unicameral) house without taking specialized rules for minority quotas or electoral thresholds into account[15]. Other systems include two-round and run-off systems (USA cf. Example 2.12, France etc.), preferential voting, transferable votes, and non-democracies without elections .

[15]If one tries to use the size of the remainders, one can run out of remainders.

[16]The map is constructed with data from a variety of sources, including (but not limited to) national election laws such as [39, 7, 50, 51], already compiled tables of electoral systems such as [40, 27, 52]. The biggest challenges in constructing such a map are the ever-evolving electoral systems used around the world and the need to compare and verify those mentioned in the literature to ensure they were not outdated. This map will also become outdated soon.

5. Multi- and two-tier systems

The impossibility theorem applies to all electoral systems, but two-tier and multi-tier systems are the ones that seriously attempt to satisfy all three desiderata. These systems therefore deserve precise treatment in their own right. In the literature (e.g., [18, 45]), two-tier systems are often described based on examples and only vaguely in general terms, and as Bertrand Russell wrote: *"Everything is vague to a degree you do not realize till you have tried to make it precise..."* [42]. Since the impossibility theorem applies to all electoral systems, the following discussion is self-contained; readers primarily interested in the theorem's implications may proceed directly to Section 6. Those curious about the architecture of compensatory systems that typically experience the consequences of the impossibility theorem can continue here, where a structural description of two-tier systems is attempted. The aim is to disentangle electoral systems into natural modular components like the subroutines of computer code that an electoral official will have to implement. We will call some of these intermediate calculations electoral steps.

Definition 5.1 (Electoral step)**.** An *electoral step* $\mathtt{T}$ is a function $\mathtt{T} : \mathcal{S}_{\tilde{c}} \times \mathcal{V}_c \to \mathcal{S}$.

Any electoral system $\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ can be considered as an electoral step $\mathtt{T}(S, V) := \mathtt{ES}(V)$ that ignores its seat-matrix input. In particular, every one-tier system is an electoral step in this sense. Electoral steps serve as a conceptual framework for thinking about two- and multi-tier systems. Often each tier corresponds to an electoral step. Electoral steps also describe the rounds of two-round systems. In that case, there are two elections V_1 and V_2 and $\mathtt{ES}_{\text{2-Round}}(V_1, V_2) = \mathtt{T}(\mathtt{ES}(V_1), V_2)$. Such systems do not play a role in the rest of the paper[17].

5.1. **Multi- and two-tier systems.** Multi-tier systems are compositions of two or more electoral steps. They are built on top of a one-tier system with additional mixing. That is, the input seat matrix S is the result of a one-tier calculation. Thus, an electoral system is multi-tier if it is not one-tier and of the form:

$$\mathtt{ES}_{\text{Multi-Tier}}(V) = \mathtt{T}(\mathtt{ES}_{\text{1-Tier}}(V), V). \tag{4}$$

Two-tier systems are a special subset of multi-tier systems (built from two electoral steps) which can be computed row-wise up to amalgamation. To build intuition and because the following definition captures almost all electoral systems in practice a simpler case is discussed first.

Definition 5.2 (Simple two-tier electoral system)**.** An electoral system $\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}_1$ is *simple two-tier* if there exist a one-tier system $\mathtt{ES}_{\text{1-Tier}} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ whose internal amalgamation (cf. Definition 2.8) is A_1, an amalgamation $A_2 : \mathcal{V}_{\tilde{c}} \to \mathcal{V}_1$, and an electoral step $\mathtt{T} : \mathcal{S}_1 \times \mathcal{V}_1 \to \mathcal{S}_1$ such that

$$\mathtt{ES}_{\text{2-Tier}}(V) = \mathtt{T}\big(A_2\mathtt{ES}_{\text{1-Tier}}(V),\ A_2A_1V\big). \tag{5}$$

We emphasize that A_2 maps the electoral outcome to just a single row (the subscript 1 in $\mathcal{V}_1$), therefore the input to $\mathtt{T}$ should be interpreted as a national total of regional seats and national total of votes. This restriction will simplify the discussion in the following somewhat as we will only need to consider distribution of leveling seats nationally (and not on constituencies). Nevertheless, as the definition is arguably a mouthful, it is instructive to go back to Iceland (cf. Example 4.1) and use that example to unpack the definition. Afterwards, a graphical explanation of the definition will be given in Figure 6.

Example 5.3 (Unpacking Definition 5.2 for the Icelandic example)**.** In the 2024 Icelandic election 6 parties passed the electoral threshold and some large number c of precincts were in use. The amalgamation $A_1 : \mathcal{V}_c \to \mathcal{V}_6$ sums up the votes in each of the six constituencies (see Figure 4). Thus A_1V is a 6×6 matrix. Recall from Example 4.1 that $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}} = (\mathtt{DH}^7, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^9, \mathtt{DH}^{11})$. The seat array $\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)$ corresponds to the distribution of regional seats. In the next step, the election is summed up completely using a total amalgamation A_2, that is A_2A_1V is the list of the total national vote for each party. Similarly, $A_2\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)$ is a row where the jth entry corresponds to the jth party's total number of regional seats. The final (national) seat distribution $\mathtt{ES}^{\text{Iceland}}$ depends both on the distribution of regional seats $A_2\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V)$ and on the national vote totals A_2A_1V. Thus, the final distribution has the form of Equation (5):

$$\mathtt{ES}^{\text{Iceland}}(V) = \tilde{\mathtt{DH}}(A_2\mathtt{ES}^{\text{Iceland}}_{\text{1-Tier}}(V), A_2A_1V),$$

where the electoral step $\mathtt{T} = \tilde{\mathtt{DH}}$ is the modified D'Hondt process that distributes the seats already distributed in the seat vector first.

Other examples of simple two-tier systems include Denmark, Sweden, Norway, Slovenia, Estonia, South Africa and New Zealand.

[17] Beyond two-round systems, this framework can also model staggered elections, such as those used for the US *Senate*. In this case, the already elected members serve as the input $\mathtt{T}(S_1, V_1)$, which itself depends on the previously elected members S_1. Additionally, the definition extends to countries such as Italy, Burundi, and the Democratic Republic of the Congo, which currently have five, two, and one lifetime-appointed senators, respectively.

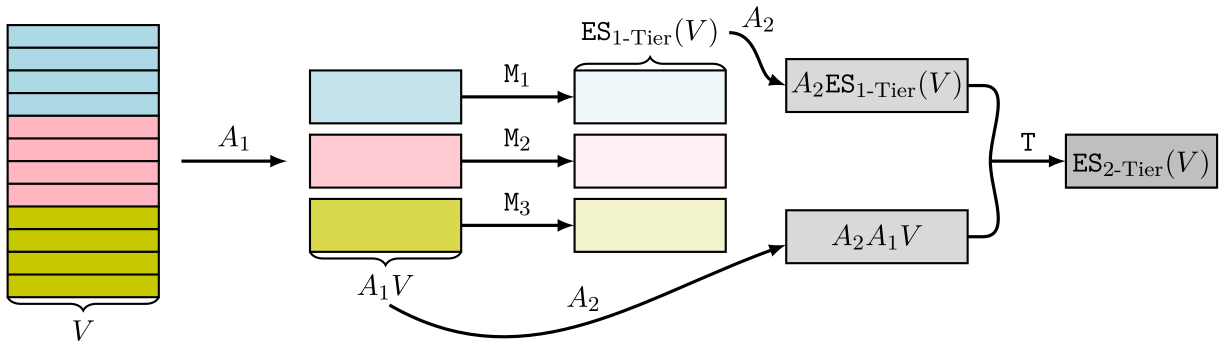


FIGURE 6. Schematic illustration of a simple two-tier system cf. Definition 5.2. The first operation corresponds to the one-tier basis of the system (see Figure 2). Both the votes and the one-tier seats are then amalgamated using A_2, which outputs only a single row. The electoral step T takes both the national vote row (A_2A_1V) and the (national) 1-tier seat distribution as input and outputs a single row of national seats.

5.2. **Parallel two-tier and multi-tier systems.** To define general two-tier systems, one should think of the system as a parallel irreducible two-tier system, similarly to the one-tier parallel electoral system of the Russian Duma in Example 2.11i). In other words, two two-tier systems that do not interact at all should still be considered two-tier.

Definition 5.4 (Two-tier electoral system)**.** An electoral system $\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}_{\hat{c}}$ is *two-tier* if there exist a one-tier system $\mathtt{ES}_{1\text{-Tier}} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$ with internal amalgamation $A_1 : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$, an amalgamation $A_2 : \mathcal{V}_{\tilde{c}} \to \mathcal{V}_{\hat{c}}$, and electoral steps $\mathtt{T}_1, \ldots, \mathtt{T}_{\hat{c}} : \mathcal{S}_1 \times \mathcal{V}_1 \to \mathcal{S}_1$, such that for each $i \in \{1, \ldots, \hat{c}\}$:

$$\mathtt{ES}_{2\text{-Tier}}(V)[i,\cdot] = \mathtt{T}_i\big((A_2S)[i,\cdot],\,(A_2A_1V)[i,\cdot]\big), \tag{6}$$

where we have set $S = \mathtt{ES}_{1\text{-Tier}}(V)$. When $\hat{c} = 1$, the system is called *simple* (cf. Definition 5.2).

As in the definition of one-tier systems Definition 2.8, this general definition of two-tier systems should be interpreted as $\hat{c}$ parallel two-tier systems. A two-tier electoral system which is not simple is the electoral system of Bosnia-Herzegovina.

Example 5.5 (Bosnia-Herzegovina's lower house of parliament *Predstavnički dom*)**.** The electoral system of the country of Bosnia-Herzegovina consists of two parallel simple two-tier systems of the form Equation (5) corresponding to the two autonomous entities, *the Federation of Bosnia and Herzegovina* and *Republika Srpska*. That is $\hat{c} = 2$.

The elections for the European parliament (cf. Example 2.12) would be an example of a two-tier system, were it not for Poland and Italy using preferential voting which does not quite fit the framework. Here every country acts as a single constituency except for Poland, Italy and Belgium. Polish and Italian elections for the European parliament would even be two-tier systems in themselves.

Finally, let us give two examples that are multi-tier but do not fit the two-tier framework.

Example 5.6 (Multi-tier systems)**.** Throughout the world some multi-tier systems are in use, see Figure 5.

(i) The Austrian lower house *Der Nationalrat*. Austria has nine federal states, which each form multi-member constituencies that are further divided into sub-constituencies, all assigned seats from the outset. First, for each federal state, the election number (i.e., the number of votes per seat, Ger. *Wahlzahl*) is calculated. Then, in each sub-constituency, seats are distributed using the largest remainders method based on the election number, rounding down. Secondly, at the state level another round of the largest remainders method, also rounded down. Finally, on the national level, additional seats are distributed using the D'Hondt method. Because interactions occur across all three levels, this system does not meet the criteria of Definition 5.4.

(ii) The Hungarian unicameral parliament *Országgylés*[18] has first-past-the-post voting in the 106 individual constituencies. Next, however, votes for unelected candidates and excess votes for elected candidates are pooled together for redistribution (similar to the abolished Italian *Scorporo* system). The remaining 93 seats are then distributed with the D'Hondt method on the pool. In Definition 5.4, only amalgamations (i.e., column summation) are allowed and not the subtractions required to construct the redistribution pool, so this system is not two-tier.

[18]Ignoring preferential minority seats.

6. Circumventing the impossibility theorem

The physicist John Bell said in a different context: "*...what is proved by impossibility proofs is lack of imagination*" [4]. In that spirit, we define a new type of electoral system that escapes the constraints of the impossibility theorem. It does not have a one-tier base, so it is mixed, but not multi-tier in the sense of (4). Afterwards, the related method of bi-proportionality is discussed. In contrast to the previous sections, these electoral systems also distribute seats on constituencies.

6.1. **Geographically ranked guaranteed proportionality.** A guaranteed proportional (with respect to a weakly proportional apportionment method that satisfies Definition 2.6) electoral system with fixed parliament size is constructed. The system will break the regionality constraint, which is in a sense obsolete in this case, since there is no preliminary allocation of seats. However, it still attains a strong form of geographic representation.

The algorithm can be described as follows: First, distribute the national seats using a weakly proportional apportionment method. This determines, right from the beginning, how many seats each party gets in the end. The goal of the method is to distribute the seats to the constituencies matching those seats to regional election outcome as closely as possible. Here is the overall idea:

- Construct rankings for each party in each constituency (e.g. the Sainte-Laguë quotients).
- Construct an overall ranking of the parliamentary seats coming from each constituency. That is, given $k_1, \dots k_c$ seats in each constituency formally consider the set of seats as the set of pairs (i, u) for $1 \le i \le c$ and $1 \le u \le k_i$. Now, prioritize the pairs ordered in some way, i.e. put on a list. The simplest way to do this is to list the quotients by size.
- Distribute seats to each party according to the quotients of the list starting from the top.

If there are no overhang seats this system is equivalent to a regional system, where the seats are first distributed regionally followed by a leveling procedure. Table 2 shows geographically ranked guaranteed proportionality deployed on the Belgian senate elections dataset that was used to evaluate bi-proportional methods in [10, 22].

Table 2. The 1981 senate election in the former Belgian province Brabant: vote distribution and seat allocation. For the overall distribution `DH` is used, and the local seats are distributed according to the `SL` quotient giving the same result as the bi-proportional methods in [10, 22]. The signs indicate the mandate change if `DH` was used both overall and locally.

	CVP	**FDF-RW**	**PVV**	**PRL**	**PS**	**SP**	**VU**	**UDRT**	**PSC**	**Ecolo**
Vote Distribution										
Br.	109,377	148,928	88,645	106,920	80,644	53,409	63,807	59,730	54,549	34,966
Le.	78,280	2,048	70,273	1,765	0	60,024	32,178	4,203	0	0
Ni.	0	17,879	707	47,579	48,965	0	923	16,984	24,590	13,060
Seat Allocation under geographically ranked guaranteed proportionality										
Br.	3^-	3^+	2	2	2	1	1	1	1	1
Le.	1^+	0	1	0	0	1	1^-	0	0	0
Ni.	0	1^-	0	1	1	0	0^+	0	0	0
Total	4	4	3	3	3	2	2	1	1	1

6.2. **Double proportionality and dual-member proportionality.** Bi-proportional systems are a class of electoral systems that circumvent the impossibility theorem, studied axiomatically by Balinski and Demange in [2]. For these systems, the regional seat distribution depends on the votes in all constituencies (as is the case with GRGP above). One such system was described by Pukelsheim in [36, Chap. 14-15] and was implemented in several cantonal electoral systems in Switzerland. Both national and constituency seat counts are fixed.

Following [29], we say that an apportionment method is *locally accountable* if there is no constituency where a locally smaller party gets more seats than a locally larger party. The proof of the impossibility theorem relied only on two properties that follow from local accountability: (1) monotonicity of local apportionment methods with a winner-take-one rule, and (2) the constraint that in a constituency with k seats, no party outside the k largest can receive a seat. The same proof also shows the following theorem even for electoral systems without a one-tier basis.

Theorem 6.1. Any locally accountable electoral system based on $c \ge 2$ constituencies that always assigns at least one local seat cannot be both guaranteed proportional and have fixed parliament size even upon assuming vote-seat consistency.

One might ask whether it is possible to weaken the local accountability assumption to winner-take-one and that is indeed possible but one needs many more parties than in the proof of the impossibility theorem[19]. This means that it is impossible to have a winner-take-one rule for both geographically ranked guaranteed proportionality and bi-proportional methods (if they are guaranteed proportional). In many practical contexts, the winner-take-one modification is essential for the legitimacy of the system[20]. Examples include the Danish island of Bornholm [16, 17], highlighted by the Danish minister of internal affairs during a 2026 parliamentary debate on the topic [20], as well as during the debate that followed the 2025 German election, which was the first without regionality [19, 6].

Another proposed system is dual-member proportionality suggested by Sean Graham for Canada [25]. The system also has a winner-take-one rule, is guaranteed proportional with respect to the Hare-LR method, and fixed parliament size. Thus, by the impossibility theorem, there must exist distributions of votes where the algorithm is not defined (and indeed this happens if one party wins more "first seats" than their Hare quota).

7. Conclusion

This paper has established a rigorous mathematical framework for electoral systems and used it to prove an impossibility theorem: no electoral system with a fixed parliament size can simultaneously guarantee both proportional representation and regionality. The result formalizes a tension that has manifested concretely across Northern European democracies this decade in the growth of the German Bundestag, the disproportionality of the 2022 Danish election, and the ongoing debates in Sweden and Norway about the adequacy of compensatory seats.

The mathematical definitions of apportionment methods, amalgamations, regionality, guaranteed proportionality and one- and two-tier systems provide a unified language for comparing electoral systems that are often discussed in isolation. Within this framework, the three desiderata (regionality, guaranteed proportionality, and fixed parliament size) are mutually incompatible whenever the number of parties is sufficiently large relative to the number of compensatory seats, as quantified by the bound in equation (3). The bound and proof of the theorem identify increasing political fragmentation as the reason that real-world electoral systems have been driven into a regime where the trade-off materializes.

The survey of real-world systems in Table 1 illustrates that electoral designers have consistently, if implicitly, navigated this trade-off by sacrificing one of the three properties. Germany prior to 2023 sacrificed fixed parliament size; Sweden and post-reform Germany sacrifice regionality; most other systems sacrifice guaranteed proportionality.

The workarounds bi-proportionality and geographically ranked guaranteed proportionality balance geographical and proportional representation and evade the impossibility theorem by not having a regional one-tier basis. Further exploration of such flexible seat allocation models that adapt dynamically to electoral outcomes, with a variable number of regional seats and compensatory seats is a natural future direction for designers of electoral systems.

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[19] If a small party is the largest in just one constituency and m (parliament size) additional parties are larger, but with spread out votes.

[20] For that reason, the 1948-1953 Danish electoral law, which was guaranteed proportional and had fixed parliament size removed a party's first seat in each constituency only as a last resort, see [16, App. C.2].

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A.1. **Impossibility theorem in the presence of thresholds.** Thresholds often act as concrete barriers to proportionality. Most commonly used are percentage thresholds, where parties below a certain percentage are excluded from (proportional) representation. Some two-tier systems have specialized rules where regional seats act as a *back door* to proportional representation. A party with a certain number of regional seats is allowed proportional representation even if the party does not meet a certain percentage threshold. Examples include Denmark, Germany[21] and New Zealand [40, p.83]. In other two-tier systems, parties failing to meet the threshold may retain their regional seats under specific conditions, but they are not granted access to compensatory seats ensuring national proportionality. An example is Sweden, which has a 4% threshold. Parties below the threshold can keep regional seats won with at least 12% of the vote regionally.

There are also examples of *hard thresholds* where parties lose regional seats if they do not make the national threshold. A notable example is Turkey, which previously had a hard electoral threshold of 10%[22].

The three different types of thresholds also highlight three different approaches to balancing proportionality and regionality in the context of thresholds. Each of these threshold types can be incorporated readily in the formalism by considering the threshold as a function on election outcomes that sets columns of parties that do not reach the threshold to zero. For simplicity, we restrict the following discussion to the case of hard thresholds.

Considering the case of a hard threshold, one could relax either the regionality (Definition 3.2) or guaranteed proportionality criteria (Definition 3.3) to only include parties above threshold and ask for a modified impossibility theorem.

If guaranteed proportionality is not relaxed, it is easy to see that the electoral system is not proportional. Similarly, if regionality is not relaxed, then a party not meeting the threshold may win a regional seat (by having its votes concentrated) and thus the electoral system does not satisfy regionality.

Now restrict attention to the case where both regionality and guaranteed proportionality are relaxed to only concern parties reaching the hard threshold. Since the threshold is hard, it puts an effective limit on the number of parties in parliament. Combining this limit with a system that has very few regional seats and a lot of compensatory seats (e.g. a huge parliament), one can construct electoral systems satisfying the two relaxed criteria as well as a fixed parliament size. For this approach to work in general (with no assumptions on the distribution of votes or regional seats), if the hard threshold is at $\alpha\%$, then in general at most $\alpha\%$ of the seats can be regional (since in the worst case a party right above the threshold could win all the regional seats with only these votes). Real-world systems using compensatory seats have far fewer compensatory seats than what is needed for this mathematical guarantee.

[21]The back door was abolished in the law of 2023, but reinstated by the constitutional court in 2024 [7].

[22]De facto thresholds are also often mentioned in the literature. They arise because obtaining representation requires at least one seat (for example $\frac{1}{150} = 0.67\%$ in the Dutch example from Example 2.10ii there are 150 seats in total). Here only superimposed thresholds are considered.

A.2. **Notation and terminology introduced in the paper.**

TABLE 3. Notation and terminology.

Formula	Name	Description	Reference
V	Election outcome	$c \times p$ matrix of vote counts; $V[i,j]$ is party j's votes in constituency i.	Definition 2.1
S	Seat distribution	$c \times p$ matrix of seat counts; $S[i,j]$ is the seats allocated to party j from constituency i.	Definition 2.2
$A : \mathcal{V}_c \to \mathcal{V}_{\tilde{c}}$	Amalgamation	Regroups the c rows of V into $\tilde{c}$ (possibly overlapping) rows.	Definition 2.4
$\mathtt{M} : \mathcal{V}_{1,p} \to \mathcal{S}_{1,p}$	Apportionment method	Turns a vote vector into a seat vector.	Definition 2.6
$\mathtt{ES} : \mathcal{V}_c \to \mathcal{S}_{\tilde{c}}$	Electoral system	Turns an election outcome on c constituencies into a seat distribution on $\tilde{c}$ constituencies.	Definition 2.3
$\mathtt{ES}_{\text{1-Tier}}(V)[i,\cdot]$ $= \mathtt{M}_i\big((AV)[i,\cdot]\big)$	One-tier system	One apportionment method per amalgamated constituency, with no interaction between them.	Definition 2.8 Figure 2
$\mathtt{T} : \mathcal{S}_{\tilde{c}} \times \mathcal{V}_c \to \mathcal{S}$.	Electoral step	Takes a seat matrix and votes and produces an updated seat matrix; the basic building block of multi-tier systems.	Definition 5.1
$\mathtt{ES}_{\text{2-Tier}}(V)[i,\cdot]$ $= \mathtt{T}_i\big((A_2S)[i,\cdot],\, (A_2A_1V)[i,\cdot]\big)$	Two-tier system	$\hat{c}$ parallel electoral steps on the one-tier seat basis $S = \mathtt{ES}_{\text{1-Tier}}(V)$.	Definition 5.4 Figure 6

SEBASTIAN TIM HOLDUM, UNIVERSITY OF COPENHAGEN, DENMARK.
Email address: pdg115@alumni.ku.dk

FREDERIK RAVN KLAUSEN, UNIVERSITY OF CAMBRIDGE, DPMMS, CAMBRIDGE, UNITED KINGDOM
Email address: frk23@cam.ac.uk