THE DISPROPORTIONATE POWER OF VOTES NEAR ELECTORAL THRESHOLDS

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Abstract. Votes for parties polling near electoral thresholds disproportionately affect parliamentary seat allocation. A derived formula quantifies the power of such votes based on the uncertainty of the party's support.

Keywords: Electoral systems, strategic voting, threshold insurance voting.

Introduction

A strategic vote is a vote that does not reflect the voter's first preference cast to achieve a specific outcome. For example, in a first-past-the-post system voters sometimes choose to vote for a party that actually has a chance to get first past the post. In this paper, we consider the power of strategic voting on smaller parties in electoral systems with a threshold - a type of strategic voting sometimes dubbed "threshold insurance voting" [3]. In many-party parliamentary systems with approximate proportional representation an electoral threshold is often introduced. The most common type of threshold is a percentage threshold, where parties below a certain percentage do not get represented.

Strategic voting was first extensively studied in majoritarian systems [1] and has only been studied in electoral systems with proportional representation in more recent decades [12, p.2]. The approaches there have often focused on assessing to what extent strategic voting takes place. For instance in Sweden, that has an electoral threshold of 4%, strategic voting is common [5, 4]; whenever a party is polling close to the electoral threshold, some voters from the same block often vote strategically to maximize the chance of their preferred coalition government. In the 2018 general election in Turkey, due to an electoral threshold of 10%, it was assessed that up to a quarter of the votes were strategic [14]. For the Danish election in 2022, there were also calls for strategic voting [7].

Given the abundance of threshold insurance voting it is natural to ask whether such strategic voting is rational? If a party is close to the threshold does it make sense to vote for it? On one hand, there is a risk that the vote is wasted. On the other, it may be essential for the parliamentary situation after the election that the party reaches the electoral threshold. If the electoral threshold is 2% does it make sense to say that a vote for a party which is polling at 1.8% counts as much as 3 votes on a larger party towards getting a certain majority?

These questions are intricate since the answers are highly dependent on the uncertainty of the polling. In the example above, if you are 99 % sure that the party at 1.8 % will end between 1.7% and 1.9% then the vote is almost surely wasted, but if the election outcome is very uncertain the picture is less clear.

This paper addresses these questions using mathematical analysis. First the setup and definition of the vote weight is introduced. Then, for parliamentary systems with a percentage threshold, a formula for the relative weight of votes on parties close to electoral threshold based on opinion polls is derived. This formula provides mathematical justification for threshold insurance voting.

In the context of US presidential elections, the related concept of voting power is studied. Here one uses opinion polls to investigate how much more power a vote in one (swing) state matters compared to one in another (non-swing) state [2].

Definition and derivation

The complete results of a free election are never known beforehand and is therefore best described probabilistically: For parliamentary elections with p parties, then their percentages $X_1, \dots, X_p$ are (correlated) random variables. After the election, these random variables are realized. For simplicity we only consider one-tier proportional electoral systems. In that case, the percentages are transformed into seats using an apportionment method, such as a remainder or quota method [9]. The presence of apportionment methods, electoral thresholds and correlation between the party percentages, could in principle lead to a very complicated relationship between the expected percentage and the expected number of seats in parliament.

If $a : [0,1]^p \to [0,1]^p$ is the apportionment function, which converts percentages of votes into percentages of seats, then the expected number of seats for the first party is $\mathbb{E}[a(X_1, \dots, X_p)_1]$, where the subscript 1 denotes the first element of the (seat)-tuple $a(X_1, \dots, X_p)$. Since this expression is difficult to analyze analytically we

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will instead analyze the problem based on proximity to proportionality. Therefore, for a threshold $t \in [0,1]$, we use the following approximation, which is central for our further inquiry,

$$\mathbb{E}[a(X_1,\dots,X_p)_1] \approx \mathbb{E}[X_1 \mathbb{1}_{\{X_1 \geq t\}}], \tag{1}$$

where $\mathbb{1}_{\{X_1 \geq t\}}$ is the indicator function that first party is above the threshold, which means that the function is 1 if the party is above the threshold and 0 if it is below the threshold. The intuition of the approximation is to substitute the true expected number of seats by the "fair share" given that it is above the threshold. In the next section we discuss the validity of this approximation. The approximation is bad if the electoral system is not close to proportional or there are many parties that will not get enough votes to pass the threshold (for instance if t is very large, as is for example the case in Turkey).

To analyze the relative weight of the vote, it would be useful to know the total number of votes N. However, as N is (usually) not known it is a random variable, but we consider it fixed and derive bounds in the limit $N \to \infty$.

Now, if we let V denote the stochastic variable that determines the total number of votes on the party in question. Hence the proportion of votes with one extra vote is $\frac{V+1}{N}$, as to be compared to $\frac{V}{N}$ without that extra vote[1]. Often potential strategic voters will have access to an opinion poll that claims for instance that $\frac{V}{N} \sim \mathcal{N}(\mu,\sigma^2)$ for some mean μ and standard deviation σ.

Usually it is unlikely, that a single additional vote ensures that the party passes the threshold. So we should compare the expected proportion of the seats in parliament to the same situation for a very large party V_L. Thus, under the approximation in (1), we arrive at the following expression for the relative weight of vote,

$$\frac{\mathbb{E}[\frac{V+1}{N}\mathbb{1}_{\{\frac{V+1}{N}\geq t\}}] - \mathbb{E}[\frac{V}{N}\mathbb{1}_{\{\frac{V}{N}\geq t\}}]}{\mathbb{E}[\frac{V_L+1}{N}\mathbb{1}_{\{\frac{V_L+1}{N}\geq t\}}] - \mathbb{E}[\frac{V_L}{N}\mathbb{1}_{\{\frac{V_L}{N}\geq t\}}]} = \frac{\text{"expected additional seat proportion by voting for party"}}{\text{"expected additional seat proportion by voting for large party"}}.$$

The denominator can be calculated. Since the party is very large, we assume that it is always above the threshold and thus the denominator equals,

$$\mathbb{E}\left[\frac{V_L+1}{N}\right] - \mathbb{E}\left[\frac{V_L}{N}\right] = \frac{1}{N}.$$

Thus, for approximately proportional systems (without many small parties below threshold) we take the following expression as the definition of W,

$$W = N\left(\mathbb{E}\left[\frac{V+1}{N}\mathbb{1}_{\{\frac{V+1}{N}\geq t\}}\right] - \mathbb{E}\left[\frac{V}{N}\mathbb{1}_{\{\frac{V}{N}\geq t\}}\right]\right) = \mathbb{E}\left[(V+1)\mathbb{1}_{\{\frac{V+1}{N}\geq t\}}\right] \tag{2}$$

For the electoral system that are our main examples of interest the number of voters N is very large. Thus, we are interested in determining W in the limit $N \to \infty$.

Proposition 1. *Suppose that the weight of vote W is defined by (2). Let $V/N \sim \mathcal{N}(\mu,\sigma^2)$. Then,*

$$W \longrightarrow t \cdot \frac{e^{-\frac{(t-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} + 1 - \Phi_{\mu,\sigma}(t), \;\; \textit{as } N \longrightarrow \infty.$$

Here $\Phi_{\mu,\sigma}$ is the cumulative distribution function of the normal distribution with parameters μ and σ^2.

Proof. By definition,

$$W = N\mathbb{E}\left[\frac{V}{N}(\mathbb{1}_{\{\frac{V+1}{N}>t\}} - \mathbb{1}_{\{\frac{V}{N}>t\}})\right] + \mathbb{P}\left[\frac{V+1}{N} > t\right] = N\mathbb{E}\left[\frac{V}{N}\mathbb{1}_{\{t-\frac{1}{N}<\frac{V}{N}<t\}}\right] + \mathbb{P}\left[\frac{V+1}{N} > t\right].$$

By continuity, as $N \to \infty$,

$$N\mathbb{E}\left[\frac{V}{N}\mathbb{1}_{\{t-\frac{1}{N}<\frac{V}{N}<t\}}\right] = N\frac{1}{\sqrt{2\pi\sigma^2}}\int_{t-\frac{1}{N}}^{t} y e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy \to \frac{t}{\sqrt{2\pi\sigma^2}}e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

as well as

$$\mathbb{P}\left[\frac{V+1}{N} > t\right] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{t-\frac{1}{N}}^{\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy = 1 - \Phi_{\mu,\sigma}(t-\frac{1}{N}) \to 1 - \Phi_{\mu,\sigma}(t).$$

□

[1] If the extra vote does not come from another party one could argue that the correct number is $\frac{V+1}{N+1}$. This difference is negligible for the analysis that follows.

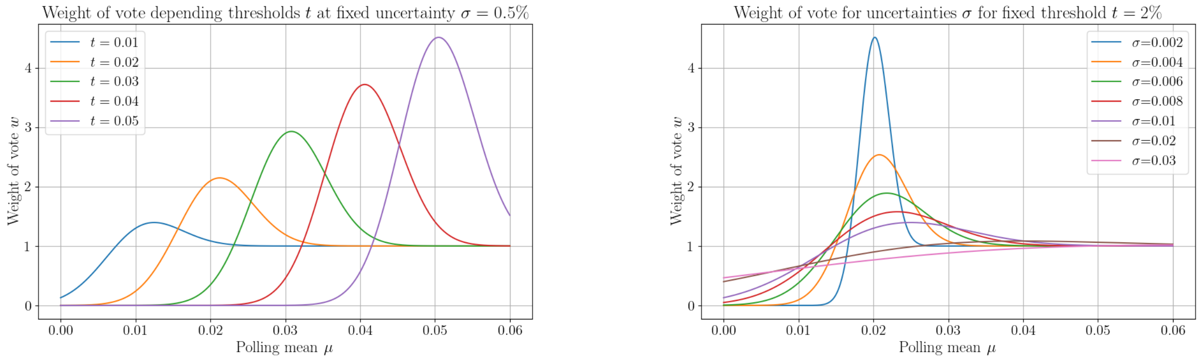

FIGURE 1. Plots of different versions the weight of the votes as a function of the mean opinion poll, that is the function $\mu \mapsto w(\mu, \sigma, t)$, where μ is the poll mean, σ the standard deviation and t the threshold. Left) We fix $\sigma_0 = 0.005$ and plot w as a function of μ for different values of t. Right) We fix $t = 0.02$ and vary σ.

With the proposition in mind define the weight function

$$w(\mu, \sigma, t) = \frac{t}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} + 1 - \Phi_{\mu,\sigma}(t). \tag{3}$$

In Figure 1, we have plotted the function w for different values of μ, σ, and t. From the plots we draw the following conclusions:

- The power of the vote increases close to the threshold and peaks for mean values very close to the threshold. For $\mu = t$ we get

$$w(t, \sigma, t) = \frac{t}{\sqrt{2\pi\sigma^2}} + \frac{1}{2}. \tag{4}$$

- Increasing the threshold increases the power of the vote (that is the height of the peak) linearly in t for fixed σ as can be seen in Figure 1 left)
- Increasing the uncertainty σ decreases the power of the threshold insurance votes.
- For common values of thresholds and statistical errors in (weighted averages of) opinion polls the peak relative weight of the vote is between a factor of 2 and 4. But it can be below 1 for parties polling at the threshold if the uncertainty is sufficiently large. Indeed, by (4), the power is less than one if and only if $t \le \sigma\sqrt{\pi/2}$. This is best understood in the limit where the uncertainty is very large. A party polling with mean t has 50% chance to be above threshold, and the probability that it is close to the threshold is very small. Note also that the assumption of normal distribution is challenged in this parameter range.

Let us make some further comments about the limits of the function. Notice that for fixed t, σ if μ is much larger than t then w is 1. It also holds that for fixed μ, t then $w \to \delta(\mu - t) + \mathbb{1}_{\{\mu > t\}}$ as $\sigma \to 0$ where δ is the Dirac delta function. This makes sense, since if it becomes more and more certain whether the vote tips the balance then the vote is useless if $\mu < t$, it has infinite relative weight if it tips the balance and has relative weight 1 if the party is above the threshold. Compare also this behavior with Figure 1.

Scaling of the weight of threshold insurance vote at the threshold. Opinion polls are based on survey samples. In the absence of more advanced techniques—such as stratification—the standard deviation is $\sigma = \sqrt{\frac{\mu(1-\mu)}{n}}$, where n is the number of respondents and μ is the mean[2] [16, Ex. 11.11]. Note that this reflects only the statistical error; sampling bias often results in larger overall errors. However, it is the case that the error increases with the size of the party. Inserting the expression for the weight of the vote for a party polling at threshold (3) yields,

$$w\left(t, \sqrt{\frac{t(1-t)}{n}}, t\right) = \frac{1}{2} + \sqrt{\frac{t}{2\pi(1-t)}}\sqrt{n}. \tag{5}$$

In Figure 2 we plot this function for different values of n. The scaling in t in the relevant regime of small t is linear.

[2]We caution the reader that often polling errors are reported as 95% confidence intervals, which are $\pm 1.96\sigma$ rather than $\pm\sigma$.

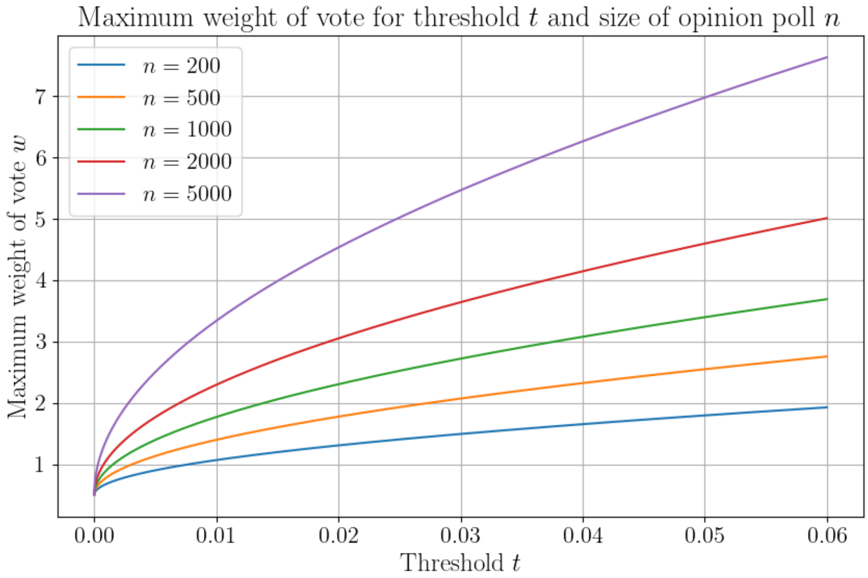

FIGURE 2. Weight of the vote as a function of threshold believing an unbiased opinion poll with sample size n based on the formula (5). When the size of the opinion poll is larger the uncertainty is smaller and so the peak for the weight of the vote in right plot in Figure 1 is sharper.

Weight of the vote in recent elections based on opinion polling. Looking only at a single opinion poll, neglecting the biases (that arguably should contribute more to the uncertainty than the statistical error) as well as advanced polling techniques we can compute the weight by inserting the standard deviation $\sqrt{\frac{\mu(1-\mu)}{n}}$ into (3). In Table 1 an overview of relevant examples is given.

Country	Year	Threshold t	Party	Poll μ	Sample Size n	Pollster	Weight
Germany	2025	5%	Linke	7.5%	2005	INSA	1.0004
Germany	2025	5%	FDP	5%	2005	INSA	4.6
Germany	2025	5%	BSW	5%	2005	INSA	4.6
Denmark	2022	2%	DF	2.9%	4577	Voxmeter	1.004
Turkey	2015	10%	HDP	9%	3850	ORC	0.84
Turkey	2015	10%	HDP	12.5%	4860	Gezici	1.000008
Sweden	2014	4%	FI	3.5%	1200	SKOP	2.1
Sweden	2014	4%	KD	4.6%	1200	SKOP	2.5

TABLE 1. Examples of the weight of vote for the last opinion poll(s) before some recent elections. The numbers are computed assuming that the polls were representative samples. The relevant references for the opinion polls are [6, 15, 8, 13].

MODEL CROSS CHECK

To do a cross check of the model we again take an opinion poll as the starting point. Suppose that there are p parties and let $X_1, \ldots, X_p$ be the random variables such that $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, where μ_i is the reported mean and σ_i the reported statistical error. Then, we sample from each of these normal distributions and normalize afterwards. This is our method for generating elections. Using these generated election outcomes we distribute the seats according to either Hare-LR or D'Hondt method.

A proxy for the weight of the vote, can be calculated by keeping the poll fixed for all parties except for one and increase the polling mean μ of that party (since we only care about small parties, we can omit the error in the renormalization). Then we iterate and use the iterations to compute the average number of seats (and the standard deviation of this number). Notice that if the opinion poll mean μ is safely above the threshold the expected number of seats is linear in μ as can be seen in Figure 3 left). Now, choosing to vote for a specific

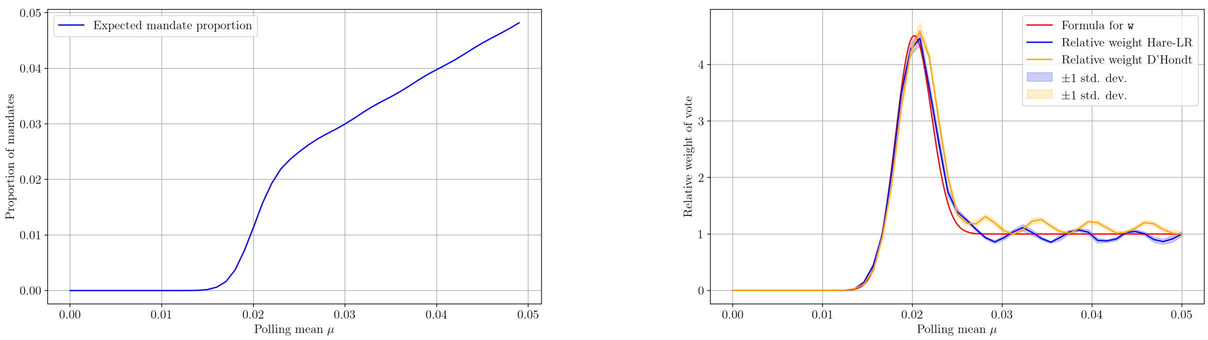

Figure 3. Left) Numerical simulation for the average proportion of seats given a specific polling mean μ. Right) We discretely differentiate the average number of seats (computed using both the D'Hondt method and the Hare-LR method). We compare the result to the function w. Details: We used the weighted polling average for national elections in Denmark from Oct, 26, 2022 with the party "Alternativet" as our party of variation[3] with fixed standard deviation $\sigma = 0.002$. This was also the value we used in the formula for w.

party corresponds to increasing the mean slightly so the weight of the vote should correspond to the slope of the average number of seats. Thus, we differentiate the average number of seats (discretely). That corresponds to the weight of the vote. In Figure 3 right)[3] we compare this to the computed function w. The curves in Figure 3 right) match well. Thus, it seems that our assumptions of looking only at one party is well justified in this specific case (which was the cause of the call to action [7]). Additionally, the model is robust to changing the apportionment method from Hare-LR to D'Hondt.

Discussion and outlook

Our quantitative perspective for relative weight of votes opens the doors for many new topics of investigation. First, the perspective opens for several possible extensions. In some countries, e.g. Denmark and Germany, there are back doors to proportional representation [10, p.83], where some number of regionally obtained seats entitle parties below the threshold to proportional representation. It could be interesting to calculate the effective weight of a vote in the presence of back doors as votes in certain constituencies may have much larger weights than for the overall systems reported here.

Here we have found a way to do the analysis based on a single party, which does not interact with other parties. But often a strategic vote is cast in an attempt to increase the chances of certain majorities [12] and in that case it may no longer be the best strategy to maximize the effective seat-weight of the vote[4].

Second, one may question the errors of opinion polls. Polls report statistical errors, but due to lack of representativity the real error is larger [11]. Quantifying this better, e.g. by comparing opinion polls to election results or by using weighted averages, would inform the results of this paper by adjusting the corresponding standard deviations.

For further progress on more mathematical side, one may try find assumptions where the approximation made in Equation (1) can be justified rigorously and quantify the error made in that approximation under suitable assumptions. One could also improve the statistical analysis modeling the elections using Dirichlet distribution and doing the calculation in Proposition 1 for the resulting marginal beta-distribution (see Appendix) - similarly, one could generate elections using a multinomial distribution. We have not done that here to simplify the presentation.

Finally, one may ask what the results presented here mean for electoral systems design. From one point of view, the effect that we have described here, that votes on parties close to the electoral threshold have larger weight is unfair as it give parties close to the threshold an artificial boost. Thus, electoral system designers may think about how to mitigate it. The effect is for instance much smaller without an electoral threshold such as in

[3]The data we used was the weighted average computed by Erik Gahner Larsen: In parts per thousand. $\mu = (A:268, B:46, C:97, D:50, F:83, I:50, K:5, O:27, V:130, Ø:69, Å:17, FG:6, M:60, Æ:89)$ and $\sigma = (A:10, B:5, C:9, D:6, F:3, I:4, K:2, O:5, V:8, Ø:4, Å:2, Q:2, M:8, Æ:9)$. We used 10 samples of 10000 iterations each.

[4]To see that, imagine that a coalition of parties are rather likely to get a majority without a party that is below but close to the electoral threshold in the poll. Then it may still be the more efficient to vote for a larger party which is sure to pass the threshold than for the smaller party even though that vote would count more.

the Netherlands. Alternatively, opinion polls could be banned (in some time frame up to the election). Finally, electoral thresholds could be softened in which case the curve would be less steep for a larger interval.

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Mathematical Appendix

One may argue that it was unnatural to model a proportion of votes which is always between 0 and 1 by normal distribution that has mass both below 0 and above 1. We did this to keep the mathematical prerequisites at a minimum for a social science reader with some familiarity with normal distributions. But as a result one should use the formula obtained in Proposition 1 with some caution for large values of σ. Let us now give some further mathematical justification.

Suppose that an opinion poll with n respondents have a mean $\mu = np_i$ for party i. Formally, to insure that proportions are always between 0 and 1 the poll can be described with a Dirichlet distribution and its marginal on the i'th party is then Beta-distributed. Hence, the number votes V on party i divided by the total number of votes N satisfies

$$\frac{V}{N} \sim \mathrm{Beta}(np_i, n(1-p_i)).$$

The weight of the vote can again be defined using (2) and the two terms be estimated in a similar fashion as above:

$$N\mathbb{E}\left[\frac{V}{N}\mathbb{1}[t-\frac{1}{N}<\frac{V}{N}<t]\right] = \frac{N}{B(np_i, n(1-p_i))}\int_{t-\frac{1}{N}}^{t} x x^{np_i}(1-x)^{n(1-p_i)}dx \to t\frac{t^{np_i}(1-t)^{n(1-p_i)}}{B(np_i, n(1-p_i))},$$

where $B(\alpha, \beta)$ is the Beta-function. and the second term:

$$\begin{aligned}\mathbb{P}\left[\frac{V+1}{N}>t\right] &= \mathbb{P}\left[\frac{V}{N}>t-\frac{1}{N}\right] = \frac{1}{B(np_i, n(1-p_i))}\int_{t-\frac{1}{N}}^{1} x^{np_i}(1-x)^{n(1-p_i)}dx \\ &= 1 - I_{t-\frac{1}{N}}(np_i, n(1-p_i)) \to 1 - I_t(np_i, n(1-p_i))\end{aligned}$$

where $I_t(\alpha, \beta)$ is the regularized incomplete Beta function, that is the cumulative distribution of the beta function. Thus, the weight of the vote in this setup converges as $N \to \infty$

$$w \to \frac{t^{np_i}(1-t)^{n(1-p_i)}}{B(np_i, n(1-p_i))} + 1 - I_t(np_i, n(1-p_i)).$$

Since n is usually large, whenever p_i is not too close to 0 or 1, the normal approximation to the beta distribution $\mathrm{Beta}(\alpha, \beta) \approx \mathcal{N}\left(\frac{\alpha}{\alpha+\beta}, \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right)$ is good. We get that $\frac{V}{N} \sim \mathcal{N}\left(p_i, \frac{p_i(1-p_i)}{n+1}\right)$, which is to be compared with the multinomial distribution which has mean np_i and variance $np_i(1-p_i)$ and thus the proportion has mean p_i and variance $\frac{p_i(1-p_i)}{n}$. In conclusion, we have sketched how the analysis extends to the more realistic case where our knowledge about the election is described with a Dirichlet distribution (with Beta marginals). This could be used as the starting point for a quantitative justification of (1).

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